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ON ZEROS OF SOME ANALYTIC FUNCTIONS RELATED TO THE RIEMANN ZETA-FUNCTION

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Abstract. For some classes of functions F, we obtain that the function $F(\zeta(s))$, where $\zeta(s)$ denotes the Riemann zeta-function, has infinitely many zeros in the strip $\frac{1}{2} < \operatorname{Re} s < 1$. For example, this is true for the functions $\sin\zeta(s)$ and $\cos\zeta(s)$.

1. Introduction

The zero-distribution of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is of particular interest in analytic number theory. It is well known that $s = -2m$, $m \in \mathbb{N}$, are so called trivial zeros of $\zeta(s)$. Moreover, $\zeta(s) \neq 0$ for $\sigma \geq 1$, and for $\sigma \leq 0$, $t \neq 0$, however, $\zeta(s)$ has infinitely many complex (non-trivial) zeros in the critical strip $\{s \in \mathbb{C} : \ 0 < \sigma < 1\}$. The famous Riemann hypothesis (RH) says that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$, and this is equivalent to the assertion that $\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$. At the moment, it is known ([1]) that at least 41 percent of all non-trivial zeros of $\zeta(s)$ in the sense of density lie on the critical line. By numerical calculations [2], the 10^{13} first non-trivial zeros are located on the line $\sigma = \frac{1}{2}$. This supports RH.

The best known result on zero-free regions for $\zeta(s)$ is of the form: there exists an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ in the region

$$\sigma \geq 1 - \frac{c}{(\log(|t|+2))^{\frac{2}{3}}(\log\log(|t|+2))^{\frac{1}{3}}}.$$

For the number $N(T)$ of all zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \beta < 1$ and $0 < \gamma \leq T$, the von Mangoldt formula

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi \mathrm{e}} + \mathrm{O}(\log T)$$

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is true. These and other classical results on zero-distribution of $\zeta(s)$ can be found in the monograph [4].

On the other hand, there exists zeta-functions similar to $\zeta(s)$ for which the Riemann hypothesis is not true. The simplest example of such functions is the Hurwitz zeta-function $\zeta(s,\alpha)$ with parameter α, $0<\alpha\leq 1$, defined, for $\sigma>1$, by

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. However, the functions $\zeta(s)$ and $\zeta(s,\alpha)$ are similar only by their definition by Dirichlet series, and in fact differ one from another very much. The function $\zeta(s,\alpha)$, except for the values $\alpha=1$ $(\zeta(s,1)=\zeta(s))$ and $\alpha=\frac{1}{2}$ $\big(\zeta(s,\frac{1}{2})=(2^s-1)\zeta(s)\big)$, does not have Euler product over primes, and this has a large influence for its properties. The main difference in the zero-distribution problem is that the function $\zeta(s,\alpha)$, $\alpha\neq 1,\frac{1}{2}$, differently from $\zeta(s)$, has zeros in the half-plane $\{s\in\mathbb{C}:\ \sigma>1\}$, and if α is transcendental or rational $\alpha\neq 1,\frac{1}{2}$, then $\zeta(s,\alpha)$ has infinitely many zeros lying in the strip $\{s\in\mathbb{C}:\ \frac{1}{2}<\sigma<1\}$. More precisely, Theorem 8.4.7 of [6] and [9, Theorem 8, p. 96], says that, for every σ_1,σ_2, $\frac{1}{2}<\sigma_1<\sigma_2<1$, there exists a constant $c=c(\alpha,\sigma_1,\sigma_2)>0$ such that, for sufficiently large T, the function $\zeta(s,\alpha)$ has more than cT zeros in the rectangle $\{s\in\mathbb{C}:\ \sigma_1<\sigma<\sigma_2,\ 0<t<T\}$.

The aim of this note is to present some examples of functions $F(\zeta(s))$ for which RH is not true. This is motivated by a better understanding of the RH problem.

For a region G on the complex plane $\mathbb{C}$, denote by $H(G)$ the space of analytic functions on G endowed with the topology of uniform convergence on compacta. Let $D=\{s\in\mathbb{C}:\ \frac{1}{2}<\sigma<1\}$. Define several classes of functions F.

1° We say that the function $F:\ H(D)\to H(D)$ belongs to the class $Lip(\beta)$, $\beta>0$, if the following hypotheses are satisfied:

a) For every polynomial $p=p(s)$ and every compact subset $K\subset D$ with connected complement, there exists an element $g\in F^{-1}\{p\}\subset H(D)$ such that $g(s)\neq 0$ on K;

b) For every compact subset $K\subset D$ with connected complement, there exist a constant $c>0$ and a compact subset $K_1\subset D$ with connected complement such that

$$\sup_{s\in K}|F(g_1(s))-F(g_2(s))|\leq c\sup_{s\in K_1}|g_1(s)-g_2(s)|^{\beta}$$

for all $g_1,g_2\in H(D)$.

Clearly, the set $\{Lip(\beta):\ \beta>0\}$ is non-empty. For example, the function $F:\ H(D)\to H(D)$, $F(g)=g'$, $g\in H(D)$, is an element of the class $Lip(1)$. This is a simple exercise of using the Cauchy integral formula.

2° Let $S = \{g \in H(D) : \ g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\}$. Denote by U the class of continuous functions $F : \ H(D) \to H(D)$ such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty.

We note that the hypothesis that the set $(F^{-1}G) \cap S \neq \varnothing$ for every open set G is theoretical and with difficulty checked. It can be replaced by a stronger but simpler one.

3° Denote by U_p the class of continuous functions $F : \ H(D) \to H(D)$ such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty.

An application of the Mergelyan theorem on the approximation of analytic functions by polynomials ([7], see also [10]) shows that $U_p \subset U$.

4° The main property of the set S is a non-vanishing of functions $g \in H(D)$. The definition of the class U_p involves polynomials, however, in the non-bounded region D, it is not easy to derive an information on the non-vanishing for the functions $g \in F^{-1}\{p\}$ with a given polynomial $p = p(s)$. Therefore, for $V > 0$, we define a bounded region $D_V = \{s \in \mathbb{C} : \ \frac{1}{2} < \sigma < 1, \ |t| < V\}$, and in place of the set S, take $S_V = \{g \in H(D_V) : \ g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\}$.

Denote by $U_{p,V}$ the class of continuous functions $F : \ H(D_V) \to H(D_V)$ such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is non-empty.

It is easily seen that, for some functions F and each polynomial $p = p(s)$, there exists a polynomial $p_1 = p_1(s) \in F^{-1}\{p\}$ and $p_1(s) \neq 0$ for $s \in D_V$. For example, this holds for the function $F(g) = c_1 g^{(1)} + \cdots + c_r g^{(r)}, \quad g \in H(D_V), \quad c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\}$.

5° For $a_1, \ldots, a_r \in \mathbb{C}$ and $F : \ H(D) \to H(D)$, let $H_{a_1,\ldots,a_r;F(0)}(D) = \{g \in H(D) : \ (g(s) - a_j)^{-1} \in H(D), \ j = 1, \ldots, r\} \cup \{F(0)\}$.

Denote by $U_{a_1,\ldots,a_r;F(0)}$ the class of continuous functions $F : \ H(D) \to H(D)$ such that $F(S) \supset H_{a_1,\ldots,a_r;F(0)}$.

The function $F(g) = g^N + a$, $N \in \mathbb{N}$, $a \in \mathbb{C}$, clearly, is an element of the class $U_{a;a}$. The functions $F(g) = \sin g$ and $F(g) = \sinh g$ belong to the class $U_{-1,1;0}$ while the functions $F(g) = \cos g$ and $F(g) = \cosh g$ are elements of the class $U_{-1,1;1}$. To see this, it suffices to solve the equation $F(g) = f$, $f \in H(D)$, in $g \in S$.

6° Denote by $\hat{U}$ the class of continuous functions $F : \ H(D) \to H(D)$ such that $s - a \in F(S)$ for every $a \in (\frac{1}{2}, 1)$.

For example, the function $F(g) = gg'$, $g \in H(D)$, belongs to the class $\hat{U}$. To see this, we have to solve the equation

$$gg' = s - a.$$

Obviously, the latter equation implies

$$(g^2)' = 2s - 2a,$$

and

$$g^2 = s^2 - 2as + C,$$

$$g = \pm\sqrt{s^2 - 2as + C}$$

with arbitrary constant C. We can choose C such that $s^2 - 2as + C \neq 0$ for $s \in D$. Thus, there exists $g \in H(D)$ satisfying the equation $F(g) = s - a$. Now we state the theorems on zeros of $F(\zeta(s))$.

Theorem 1.1. *Suppose that F belongs to at least one of the classes $Lip(\beta)$, U, U_p, $U_{p,V}$ and $\hat{U}$. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, F) > 0$ such that, for sufficiently large T, the function $F(\zeta(s))$ has more than cT zeros lying in the rectangle $\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2,\ 0 < t < T\}$.*

Theorem 1.2. *Suppose that F is an element of the class $U_{a_1,\dots,a_r;F(0)}$, where $\operatorname{Re} a_j \notin (-\frac{1}{2}, \frac{1}{2})$, $j = 1, \dots, r$. Then the same assertion as in Theorem 1.1 is true.*

Proof of Theorems 1.1 and 1.2 is based on the universality of $F(\zeta(s))$.

2. Universality of $F(\zeta(s))$

In [8], S. M. Voronin discovered a very interesting approximation property of the function $\zeta(s)$ which now is called universality. He proved that any analytic non-vanishing function can be approximated with a given accuracy uniformly on compact subsets of the strip D by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. More precisely, let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\} > 0.$$

Here $\text{meas}\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Since the approximated functions are non-vanishing, the Voronin theorem does not give any information on the number of zeros of $\zeta(s)$ in the strip D. In [5], we began to consider universality theorems for $F(\zeta(s))$ in which the shifts $F(\zeta(s + i\tau))$ approximate not necessarily non-vanishing analytic functions. Thus, theorems of such a kind provide an information on zeros of $F(\zeta(s))$. For the proof of Theorems 1.1 and 1.2, we apply the following universality properties of $F(\zeta(s))$.

Lemma 2.1. *Suppose that the function F satisfies the hypotheses at least one of the classes $Lip(\beta)$, U and U_p. Let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\{\tau \in [0, T] : \sup_{s\in K} |F(\zeta(s + i\tau)) - f(s)| < \varepsilon\} > 0.$$

PROOF. The case of U was considered in [5]. Since $U_p \subset U$, it remains to prove the lemma for the class $Lip(\beta)$. By the Mergelyan theorem, there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{2.1}$$

Let $g \in F^{-1}\{p\}$ and $g(s) \neq 0$ on K. By the Voronin theorem, the set of $\tau \in \mathbb{R}$ such that

$$\sup_{s\in K_1} |\zeta(s + i\tau) - g(s)| < c^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}$$

has a positive lower density. This and 2° of the class $Lip(\beta)$ show that the set of $\tau \in \mathbb{R}$ such that

$$\sup_{s\in K} |F(\zeta(s + it)) - p(s)| < \frac{\varepsilon}{2}$$

also has a positive lower density what together with (2.1) proves the lemma. □

LEMMA 2.2. *Let K and $f(s)$ be the same as in Lemma 2.1. Suppose that $V > 0$ is such that $K \subset D_V$, and that $F \in U_{p,V}$. Then the same assertion as in Lemma 2.1 is true.*

The lemma in a bit different form is given in [5].

LEMMA 2.3. *Suppose that the function $F \in U_{a_1,\dots,a_r;F(0)}$. If $r = 1$, let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous and $\neq a_1$ function on K which is analytic in the interior of K. If $r \geq 2$, let $K \subset D$ be an arbitrary compact subset and $f \in H_{a_1,\dots,a_r;F(0)}(D)$. Then the same assertion as in Lemma 2.1 is true.*

PROOF. The lemma for $r = 1$ was proved in [5], therefore, we consider the case $r \geq 2$, only. We use a probabilistic limit theorem for $F(\zeta(s))$. Denote by $\mathcal{B}(H(D))$ the σ-field of Borel sets of the space $H(D)$. Then is known ([4]) that $T^{-1}\text{meas}\{\tau \in [0, T] :\ \zeta(s + i\tau) \in A\}$, $\quad A \in \mathcal{B}(H(D))$, converges weakly to the probability measure P_ζ on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$, and the support of P_ζ is the set S. This and the continuity of F implies that

$$\frac{1}{T}\text{meas}\{\tau \in [0, T] :\ F(\zeta(s + i\tau)) \in A\}, \quad A \in \mathcal{B}(H(D)), \tag{2.2}$$

converges weakly to $P_\zeta F^{-1}$ as $T \to \infty$.

Let $g \in H_{a_1,\dots,a_r;F(0)}(D)$ be arbitrary. Then we can find $g_1 \in S$ such that $F(g_1) = g$. This, the continuity of F and the above remarks show that, for every open neighbourhood G of g, the inequality $P_\zeta F^{-1}(G) > 0$ holds. This means that g belongs to the support of the measure $P_\zeta F^{-1}$. Thus, the support of $P_\zeta F^{-1}$ contains the set $H_{a_1,\dots,a_r;F(0)}(D)$, and even its closure.

Let

$$\hat{G} = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \frac{\varepsilon}{2} \right\}.$$

Since $f(s) \in H_{a_1,\dots,a_r;F(0)}(D)$ is an element of the support of $P_\zeta F^{-1}$, therefore, $P_\zeta F^{-1}(\hat{G}) > 0$. This together with weak convergence of (2.2) shows that

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\{\tau \in [0,T] : \sup_{s\in K} |F(\zeta(s+i\tau)) - f(s)| < \varepsilon\} \geq P_\zeta F^{-1}(\hat{G}) > 0.$$

∎

LEMMA 2.4. *Suppose that $F: H(D) \to H(D)$ is a continuous function, $K \subset D$ is a compact subset, and $f \in F(S)$. Then the same assertion as in Lemma 2.1 is true.*

Proof is similar to that of the case $r \geq 2$ of Lemma 2.3.

3. Proof of Theorems

PROOF OF THEOREM 1.1. We apply Lemmas 2.1, 2.2 and 2.4 with $K = \{s \in \mathbb{C} : |s - \sigma_0| \leq \rho\}$ and $f(s) = s - \sigma_0$, where

$$\sigma_0 = \frac{\sigma_1 + \sigma_2}{2} \quad \text{and} \quad \rho = \frac{\sigma_2 - \sigma_1}{2}.$$

Then the mentioned lemmas show that, for every $\varepsilon > 0$, the set of $\tau \in \mathbb{R}$ satisfying the inequality

$$\sup_{s\in K} |F(\zeta(s+i\tau)) - f(s)| < \varepsilon \tag{3.1}$$

has a positive lower density. Now we take ε such that

$$0 < \varepsilon < \inf_{|s-\sigma_0|=\rho} |f(s)| = \rho.$$

Then the functions $f(s)$ and $F(\zeta(s+i\tau)) - f(s)$ on the disc K satisfy the hypotheses of the classical Rouché theorem. Since the function $f(s)$ has one zero $s = \sigma_0$ on K, by Rouché's theorem, the sum $F(\zeta(s+i\tau))$ of the functions $F(\zeta(s+i\tau)) - f(s)$ and $f(s)$ also has one zero on K. However, the measure of $\tau \in [0,T]$ satisfying inequality (3.1), for sufficiently large T, is greater than cT, and the theorem is proved. ∎

REMARK 3.1. The hypothesis of the class $\hat{U}$ can be replaced by a more general one: there exists a function $g \in S \setminus \{0\}$ such that, for every $a \in \left(\frac{1}{2}, 1\right)$, there exists b with $F(g(b)) = 0$ and $\text{Re}\, b = a$. The proof runs in the above way with $f(s) = F(g(s))$ and $K = \{s \in \mathbb{C} : |s - \sigma_0 - it_0| \leq \rho\}$, where t_0 is such that $F(g(\sigma_0 + it_0)) = 0$.

Proof of Theorem 1.2. We preserve the notation used in the proof of Theorem 1.1. Since $\operatorname{Re} a_j \notin (-\frac{1}{2}, \frac{1}{2})$, we have that the function $f(s) = s - \sigma_0 \neq a_j$, in the strip D, $j = 1, \dots, r$. Therefore, the function $f(s)$ on the disc K satisfies the hypotheses of Lemma 2.3, and the further proof runs in the same way as that of Theorem 1.1. $\square$

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